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ON THE THEORY OF CONDUCTIVE HEAT TRANSFER IN FINITE REGIONS*

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Abstract—Finite integral transform techniques are applied to the solution of three-dimensional, transient heat-conduction problems with general time-dependent heat sources and boundary conditions. The latter includes, as special cases, the boundary conditions of the first, second and third kind (prescribed temperature, prescribed heat flux and Newtonian convection conditions) or any combination of these three, which are often encountered in engineering problems.

The solution is obtained in terms of quasi-steady and transient terms and is given in the form of infinite series. From the solution it is shown how the general problem can be reduced to several simpler ones for which solutions may usually readily be available in the literature. The paper constitutes a generalization of a recent one in which the volume and surface source functions are assumed separable in the space and time variables.

NOMENCLATURE

$A_i(s_i) \geqslant 0$, $B_i(s_i) \geqslant 0$,	boundary coefficient functions defined on S_i;
$A \geqslant 0, B > 0$,	constant boundary coefficients on circle $r = a$;
a,	circle radius;
$C(\lambda_m)$ or C_m,	coefficients defined by (7);
C_{km},	coefficients defined by (32);
$F(P)$,	initial temperature distribution function in R;
$f_i(s_i, t)$,	source functions on S_i;
$f(\varphi, t)$,	source function on circle $r = a$;
i,	$1, 2, 3, \ldots, q$;
$J_k(x)$,	Bessel function of the first kind of order k and of argument x;
j,	$0, 1, 2, \ldots, q$;
K,	thermal conductivity of R;
k,	$0, 1, 2, \ldots, \infty$;
m,	$1, 2, \ldots, \infty$;
n_i,	outward normal of S_i;
P,	point in R;
$Q(P, t)$,	internal heat generation function per unit time and per unit volume;
q,	number of co-ordinate surfaces of R;
R,	homogeneous region in P-space;
r,	radial space co-ordinate in plane circular polar co-ordinates;
S_i,	ith co-ordinate surface of R;
s_i,	point on S_i;
$T(P, t)$,	temperature distribution in R;
$T_{0j}(P, t)$,	temperature distributions defined by (11) or (26);
$T_0(r, \varphi, t)$,	temperature distribution defined by (27);
$T_1(P, t)$,	temperature distribution defined by (18) or (19);
$T_2(P, t)$,	temperature distribution defined by (24);
t,	time variable;
$V(P)$,	volume in P-space;
δ_{ij}, δ_{0j},	Kronecker delta;
$\dfrac{\partial}{\partial t}$,	partial derivative with respect to t;

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$\Theta_j(P, \tau, t)$,	temperature distributions defined by (20) or (21);
κ,	thermal diffusivity of R;
λ_m,	eigenvalues in P-space;
λ_{km},	eigenvalues in r, φ-space;
μ,	dimensionless constant defined by (31);
τ,	parameter and variable of integration;
$\phi(\lambda_m, P)$ or $\phi_m(P)$,	eigenfunctions in P-space;
$\phi_{km}(r, \varphi)$,	eigenfunctions in r, φ-space;
φ,	angular space co-ordinate in plane circular polar co-ordinates;
∇,	gradient vector in P-space;
∇^2,	Laplace operator in P-space;
$(\dot{\ })$,	$\frac{\partial(\ )}{\partial t}$ or $\frac{\partial(\ )}{\partial \tau}$;
$(\bar{\ })$, $\int_R \phi_m(P)(\ )\,dV$,	finite integral transform of $(\ )$;
$(\tilde{\ })$,	finite cosine transform of $(\ )$ defined by (34).

INTRODUCTION

In solving beam vibration problems with time-dependent boundary conditions Mindlin and Goodman [1] used a modified separation-of-variables method which was later extended by Chow [2] and Archer [3] to reduce the non-homogeneous boundary conditions to homogeneous ones. In a recent paper of this journal Ojalva [4] applied this technique to solve heat-conduction problems with time-dependent heat sources and boundary conditions. In all of these references the time-dependence of the boundary conditions and of the heat source is introduced by separable space and time functions. The purpose of this paper is to determine the temperature distributions in homogeneous, finite continua for which internal heat generation rate and boundary conditions are arbitrary functions of space and time. The method used is the well-known finite integral transform method which dates back to 1936 in its first application to the heat equation by Doetsch [5], and from that time has been utilized constantly by many researchers.

STATEMENT AND SOLUTION OF THE PROBLEM

Consider a stationary, homogeneous, isotropic region R with constant thermal properties. Let its bounding surface S be composed of continuous co-ordinate surfaces S_i, q in number, in a conveniently chosen three-dimensional co-ordinate system. The equation of heat conduction [6] is

$$\nabla^2 T(P, t) + \frac{1}{K} Q(P, t) = \frac{1}{\kappa}\frac{\partial T}{\partial t}(P, t), \quad P \text{ in } R,\ t > 0, \quad (1)$$

with the general boundary conditions

$$A_i(s_i)\frac{\partial T(P, t)}{\partial n_i} + B_i(s_i)\,T(P, t) = f_i(s_i, t), \quad P \text{ on } S_i,\ t > 0, \quad (2)$$

and the initial condition

$$T(P, t) = F(P), \ P \text{ in } R,\ t = 0. \quad (3)$$

Consider the three-dimensional Sturm–Liouville system

$$\nabla^2 \phi_m(P) + \lambda_m^2\, \phi_m(P) = 0, \quad P \text{ in } R, \quad (4a)$$

$$A_i \frac{\partial \phi_m(P)}{\partial n_i} + B_i\, \phi_m(P) = 0, \quad P \text{ on } S_i. \quad (4b)$$

With $\phi_m(P)$ as kernel, a three-dimensional finite integral transform of $T(P, t)$ is defined as

$$\bar{T}(\lambda_m, t) = \int_R \phi(\lambda_m, P)\, T(P, t)\, dV. \quad (5)$$

It follows, from the well-known theorems of completeness and orthogonality of the eigenfunctions $\phi_m(P)$ satisfying system (4), that $T(P, t)$ can be expanded into a triple infinite series as

$$T(P, t) = \sum_{m=1}^{\infty} C(\lambda_m)\, \phi(\lambda_m, P)\, \bar{T}(\lambda_m, t), \quad (6)$$

where

$$\frac{1}{C(\lambda_m)} = \int_R \phi^2(\lambda_m, P)\, dV. \quad (7)$$

(6) is sometimes called the inversion formula for (5). Applying the transform (5) to $\nabla^2 T(P, t)$

and using the Gauss Theorem in region R one gets

$$\overline{\nabla^2 T}(\lambda_m, t) = \sum_{i=1}^{q} \int_{S_i} \left[\phi_m(s_i) \frac{\partial T(s_i, t)}{\partial n_i} - T(s_i, t) \frac{\partial \phi_m(s_i)}{\partial n_i} \right] ds_i - \lambda_m^2 \bar{T}(\lambda_m, t),$$

which, on incorporating (2) and (4b) becomes

$$\overline{\nabla^2 T}(\lambda_m, t) = \sum_{i=1}^{q} \int_{S_i} \frac{\phi_m(s_i)}{A_i(s_i)} f_i(s_i, t)\, dS_i - \lambda_m^2 \bar{T}(\lambda_m, t). \qquad (8)$$

It is to be noted that if $A_i = 0$, $[\phi_m / A_i(s_i)]$ is to be replaced with $[-1/B_i(s_i)] \cdot [\partial \phi_m / \partial n_i]$ in (8) and all the results will be applicable for this case also.

The application of (5) to both sides of (1) yields, in view of (8), the following ordinary, first order, linear differential equation

$$\frac{d\bar{T}(\lambda_m, t)}{dt} + \kappa \lambda_m^2 \bar{T}(\lambda_m, t) = \frac{\kappa}{K} \bar{Q}(\lambda_m, t) + \kappa \sum_{i=1}^{q} \int_{S_i} \frac{\phi(\lambda_m, s_i)}{A_i(s_i)} f_i(s_i, t)\, dS_i. \qquad (9)$$

The solution to (9) subject to (3) transformed by (5) is

$$\bar{T}(\lambda_m, t) = \exp(-\kappa \lambda_m^2 t) \left\{ \bar{F}(\lambda_m) + \kappa \int_0^t \exp(\kappa \lambda_m^2 \tau) \left[\frac{\bar{Q}(\lambda_m, \tau)}{K} + \sum_{i=1}^{q} \int_{S_i} \frac{\phi(\lambda_m, s_i)}{A_i(s_i)} f_i(s_i, \tau)\, dS_i \right] d\tau \right\}. \qquad (10)$$

Thus, (6) used in conjunction with (7) and (10) gives the solution to the problem.

The eigenfunctions $\phi_m(P)$ and the corresponding eigenvalues λ_m are to be determined separately from (4). It is well established, in standard works on the theory of eigenvalue problems, that ϕ_m and λ_m each comprise a discrete set of infinite terms which are all real. In three-dimensional problems ϕ_m and λ_m will each be made up of three components in general. It is therefore appropriate to speak of resultant eigenfunctions and resultant eigenvalues (or eigenfunction and eigenvalue vectors) in connection with (4). The above results also apply to the diffusion equation in n-dimensional space providing the transform (5) is taken in the n-dimensional space.

It cannot be shown directly from (6) that the latter does satisfy the boundary conditions (2). This is due to the convergence problems that arise when dealing with series-form solutions. To avoid this situation it is desirable to obtain from (6) an alternate form of solution composed of quasi-steady-state and transient parts. For this purpose it is required that the temperature functions $T_{0j}(P, t)$ be solutions of the following system:

$$\nabla^2 T_{0j}(P, t) + \frac{\delta_{0j}}{K} Q(P, t) = 0, \quad P \text{ in } R, \qquad (11a)$$

$$A_i(s_i) \frac{\partial T_{0j}(P, t)}{\partial n_i} + B_i(s_i) T_{0j}(P, t) = \delta_{ij} f_i(s_i, t), \quad P \text{ on } S_i. \qquad (11b)$$

Here it is necessary to assume further that $B_i(s_i) \neq 0$ for all $i = 1, 2, \ldots, q$, simultaneously. Otherwise the solutions $T_{0j}(P, t)$ satisfying (11) do not exist, unless

$$\delta_{0j} \int_R Q(P, t)\, dV + K \delta_{ij} \int_{S_i} \frac{f_i(s_i, t)}{A_i(s_i)} dS_i = 0.$$

The application of the Gauss Theorem to system (4) shows that

$$\lambda_m^2 = C_m \left\{ \int_R [\nabla \phi_m(P)]^2\, dV + \sum_{i=1}^{q} \int_{S_i} \frac{B_i(s_i)}{A_i(s_i)} \phi_m^2(s_i)\, dS_i \right\}, \quad A_i(s_i) \neq 0, \qquad (12a)$$

$$\lambda_m^2 = C_m \int_R [\nabla \phi_m(P)]^2\, dV, \quad A_i(s_i) = 0 \text{ for all } i. \qquad (12b)$$

It follows from (12) that, excepting the case of $B_i(s_i) = 0$ for all i, which requires special attention, the values of λ_m^2 are all non-zero. Then, the application of (5) to $\sum_{j=0}^{q} \nabla^2 T_{0j}(P, t)$ yields, in view of (11),

$$\sum_{j=0}^{q} \bar{T}_{0j}(\lambda_m, t) = \frac{1}{\lambda_m^2}\left[\frac{1}{K}\bar{Q}(\lambda_m, t) + \sum_{i=1}^{q}\int_{S_i}\frac{\phi(\lambda_m, s_i)}{A_i(s_i)} f_i(s_i, t)\, dS_i\right]. \qquad (13)$$

Using (13), (10) is rewritten as

$$\bar{T}(\lambda_m, t) = \sum_{j=0}^{q} \bar{T}_{0j}(\lambda_m, t) + \exp(-\kappa\lambda_m^2 t)\{\bar{F}(\lambda_m) - \sum_{j=0}^{q} [\bar{T}_{0j}(\lambda_m, 0) + \textstyle\int_0^t \exp(-\kappa\lambda_m^2\tau)\, \dot{\bar{T}}_{0j}(\lambda_m, \tau)\, d\tau]\}. \qquad (14)$$

(6) can be written in the equivalent form

$$T(P, t) = \sum_{j=0}^{q} T_{0j}(P, t) + \sum_{m=1}^{\infty} C(\lambda_m)\,\phi(\lambda_m, P)\,[\bar{T}(\lambda_m, t) - \sum_{j=0}^{q} \bar{T}_{0j}(\lambda_m, t). \qquad (15)$$

Introducing (14) into (15) one gets

$$T(P, t) = \sum_{j=0}^{q} T_{0j}(P, t) + \sum_{m=1}^{\infty} C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\{\textstyle\int_R \phi_m(P) F(P)\, dV - \sum_{j=0}^{q} [\int_R \phi_m(P)\, T_{0j}(P, 0)\, dV + \int_0^t \exp(\kappa\lambda_m^2\tau)\int_R \phi_m(P)\, \dot{T}_{0j}(P, \tau)\, dV\, d\tau]\}. \qquad (16)$$

Here the infinite series terms contain the T_{0j} functions. An equivalent expression containing the heat source and boundary terms is obtained by introducing (13) into (16):

$$T(P, t) = \sum_{j=0}^{q} T_{0j}(P, t) + \sum_{m=1}^{\infty} C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\left\{\int_R \phi_m(P) F(P)\, dV - \frac{1}{\lambda_m^2}\left[\frac{1}{K}\int_R \phi_m(P)\, Q(P, 0)\, dV + \sum_{i=1}^{q}\int_{S_i}\frac{\phi_m(s_i)}{A_i(s_i)} f_i(s_i, 0)\, dS_i\right] - \frac{1}{\lambda_m^2}\int_0^t \exp(\kappa\lambda_m^2\tau)\left[\frac{1}{K}\int_R \phi_m(P)\, \dot{Q}(P, \tau)\, dV + \sum_{i=1}^{q}\int_{S_i}\frac{\phi_m(s_i)}{A_i(s_i)} \dot{f}_i(s_i, \tau)\, dS_i\right] d\tau\right\}. \qquad (17)$$

(17) may be preferred to (16) in cases where the evaluation of the surface integrals is less laborious than that of the volume integrals.

It can be shown, by direct substitution, that (16) and (17) satisfy the differential equation (1), the boundary conditions (2) and the initial condition (3). It should be remarked here that the uniform convergence of the infinite series in (16) or (17) is ensured by the requirements that $F(P)$, $Q(P, t)$ and $f_i(s_i, t)$ possess continuous first and second order partial derivatives in the space variables, and that $Q(P, t)$ and $f_i(s_i, t)$ possess continuous first order partial derivatives with respect to t.

REDUCTION TO SIMPLER PROBLEMS

The splitting up of heat conduction problems into several simpler ones is well-known and has been extensively used. It is interesting to see how easily the general problem treated here can be reduced to a number of simpler problems, starting from the solution (16).

Let

$$T_1(P, t) = \sum_{m=1}^{\infty} C(\lambda_m)\,\phi(\lambda_m, P)\exp(-\kappa\lambda_m^2 t)\textstyle\int_R \phi(\lambda_m, P)\,[F(P) - \sum_{j=0}^{q} T_{0j}(P, 0)]\, dV. \qquad (18)$$

Then, it is easily seen that $T_1(P, t)$ is the solution of

$$\left.\begin{aligned} &\nabla^2 T_1(P, t) = \frac{1}{\kappa}\frac{\partial T_1(P, t)}{\partial t}, \quad P \text{ in } R,\ t > 0. \\ &\text{with} \\ &A_i(s_i)\frac{\partial T_1(P, t)}{\partial n_i} + B_i(s_i)\, T_1(P, t) = 0, \\ &\qquad\qquad P \text{ on } S_i,\ t > 0, \\ &\text{and} \\ &T_1(P, t) = F(P) - \sum_{j=0}^{q} T_{0j}(P, 0), \\ &\qquad\qquad P \text{ in } R,\ t = 0. \end{aligned}\right\} \qquad (19)$$

Again, let

$$\Theta_j(P, \tau, t) = \sum_{m=1}^{\infty} C(\lambda_m)\, \phi(\lambda_m, P) \exp(-\kappa\lambda_m^2 t) \int_R \phi(\lambda_m, P)\, T_{0j}(P, \tau)\, \mathrm{d}V. \qquad (20)$$

Then, it is easily seen that $\Theta_j(P, \tau, t)$ are the solutions of

$$\left.\begin{aligned} &\nabla^2\Theta_j(P, \tau, t) = \frac{1}{\kappa}\frac{\partial\Theta_j(P, \tau, t)}{\partial t}, \quad P \text{ in } R,\ t > 0, \\ &\text{with} \\ &A_i(s_i)\frac{\partial\Theta_j(P, \tau, t)}{\partial n_i} + B_i(s_i)\Theta_j(P, \tau, t) = 0, \quad P \text{ on } S_i,\ t > 0, \\ &\text{and} \\ &\Theta_j(P, \tau, t) = T_{0j}(P, \tau), \quad P \text{ in } R,\ t = 0. \end{aligned}\right\} \qquad (21)$$

Also, it follows from (20) that

$$\sum_{j=0}^{q} \int_0^t \left[\frac{\partial\Theta_j(P, \tau', t-\tau)}{\partial\tau'}\right]_{\tau'=\tau} \mathrm{d}\tau = \sum_{m=1}^{\infty} C_m\, \phi_m(P) \exp(-\kappa\lambda_m^2 t) \left[\sum_{j=0}^{q} \int_0^t \exp(\kappa\lambda_m^2\tau) \int_R \phi_m(P)\, \dot{T}_{0j}(P, \tau)\, \mathrm{d}V\, \mathrm{d}\tau\right]. \qquad (22)$$

Using (18), (20) and (22) the solution (16) can be rewritten as

$$T(P, t) = T_1(P, t) + \sum_{j=0}^{q}\left\{T_{0j}(P, t) - \int_0^t \left[\frac{\partial\Theta_j(P, \tau', t-\tau)}{\partial\tau'}\right]_{\tau'=\tau} \mathrm{d}\tau\right\}. \qquad (23)$$

The $T_{0j}(P, t)$ functions are the steady-state solutions in which "t" plays the role of a parameter and are determined from the system (11); $T_1(P, t)$ and $\Theta_j(P, \tau, t)$ are the transient solutions determined from the completely homogeneous systems (19) and (21) respectively, in which the forcing is due to fictitious initial temperature distributions only, and where "τ" is looked upon as a parameter. For sufficiently large values of time t the transient response $T_1(P, t)$ becomes vanishingly small and the temperature distribution tends to

$$T_2(P, t) = \sum_{j=0}^{q}\left\{T_{0j}(P, t) - \int_0^t \left[\frac{\partial\Theta_j(P, \tau', t-\tau)}{\partial\tau'}\right]_{\tau'=\tau} \mathrm{d}\tau\right\}. \qquad (24)$$

The asymptotic behavior of $T_2(P, t)$ for large t is termed the quasi-steady response, although this term is usually used in cases where $Q(P, t)$ and $f_i(s_i, t)$ remain bounded as $t \to \infty$, such as the simple harmonic variation of the source functions with t.

If $Q(P, t)$ and $f_i(s_i, t)$ are time-independent, (24) becomes

$$T_2(P) = \sum_{j=0}^{q} T_{0j}(P) \qquad (25)$$

where $T_2(P)$ is the steady response.

CONCLUDING REMARKS

For regions of simple geometry the T_{0j} functions may be available in standard works when A_i and B_i are constants. Especially in one-dimensional regions most are readily obtainable in closed form. In two and three-dimensional regions, the solutions of T_{0j} can be determined in the form of single and double infinite series, respectively. The easiest method of solution seems to be the repeated use of one-dimensional finite integral transforms. The application of the three-dimensional or resultant transform (5) to the determination of $T_{0j}(P, t)$ in three-dimensional regions yields

$$T_{0j}(P, t) = \sum_{m=1}^{\infty} \frac{\phi_m(P) \int_{S_j} \phi_m(s_j, t)\, \mathrm{d}S_j}{A_j\, \lambda_m^2 \int_R \phi_m^2(P)\, \mathrm{d}V}, \quad (j = 1, 2, \ldots, q), \qquad (26a)$$

$$T_{00}(P, t) = \frac{1}{K}\sum_{m=1}^{\infty} \frac{\phi_m(P) \int_R \phi_m(P)\, Q(P, t)\, \mathrm{d}V}{\lambda_m^2 \int_R \phi_m^2(P)\, \mathrm{d}V}. \qquad (26b)$$

Instead of this triple infinite series form of solution, when the particular geometry of the region in question is given, an alternate expression in terms of double infinite series can be achieved by repeatedly applying to (11a) those one-dimensional components of (5) which do not utilize as kernels that component of the eigenvector which corresponds to the space variable in the direction of n_j. (11a) is thus converted into a linear, second order, ordinary differential equation subject to the boundary conditions (11b) transformed similarly, with A_i and B_i assumed constant. The resulting system is readily solved and upon inversion gives the solution to $T_{0j}(P, t)$ in the form of double infinite series. In other words (26) gives rise to summation formulas for certain series summed over the discrete set of infinite terms of one of the three components of the eigenvalues λ_m. To clarify this point further a simple example is in order.

Consider a two-dimensional problem inside a circle of radius a. For the sake of simplicity let $Q(P, t) = 0$. Since $q = 1$, we may drop the subscripts i and j. The system corresponding to (11), in plane circular polar co-ordinates, is given by

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) T_0(r, \varphi, t) = 0, \quad (0 \leqslant r < a;\ 0 \leqslant \varphi \leqslant 2\pi), \tag{27a}$$

and

$$A\frac{\partial T_0(r, \varphi, t)}{\partial r} + B\,T_0(r, \varphi, t) = f(\varphi, t), \quad (r = a;\ 0 \leqslant \varphi \leqslant 2\pi). \tag{27b}$$

The corresponding eigenfunctions satisfy the system

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+\lambda_{km}^2\right)\phi_{km}(r, \varphi) = 0, \quad (0 \leqslant r < a;\ 0 \leqslant \varphi \leqslant 2\pi), \tag{28a}$$

$$A\frac{\partial \phi_{km}(r, \varphi)}{\partial r} + B\,\phi_{km}(r, \varphi) = 0, \quad (r = a;\ 0 \leqslant \varphi \leqslant 2\pi). \tag{28b}$$

The solutions of (28a) well-behaved at the origin are

$$\phi_{km}(r, \varphi) = J_k(\lambda_{km} r)\begin{Bmatrix}\cos k\varphi\\ \sin k\varphi\end{Bmatrix}, \quad (k = 0, 1, 2, \ldots). \tag{29a}$$

The eigenvalues λ_{km} are the mth positive roots of

$$A\lambda_{km} J_k'(\lambda_{km} a) + BJ_k(\lambda_{km} a) = 0, \tag{29b}$$

or of

$$(k + \mu) J_k(\lambda_{km} a) = \lambda_{km}\, aJ_{k+1}(\lambda_{km} a), \tag{30}$$

where

$$\mu = \frac{aB}{A} > 0, \tag{31}$$

and the prime in (29b) denotes differentiation with respect to the argument. By (7) and (29a),

$$\left.\begin{aligned}\frac{1}{C_{km}} &= \int_0^a\int_0^{2\pi} J_k^2(\lambda_{km} r)\begin{Bmatrix}\cos^2 k\varphi\\ \sin^2 k\varphi\end{Bmatrix} r\,dr\,d\varphi\\ &= \frac{\pi}{2\lambda_{km}^2}(a^2\lambda_{km}^2 + \mu^2 - k^2) J_k^2(\lambda_{km} a), \quad k = 1, 2, \ldots\\ &= \frac{\pi}{\lambda_{0m}^2}(a^2\lambda_{0m}^2 + \mu^2) J_0^2(\lambda_{0m} a), \quad k = 0.\end{aligned}\right\} \tag{32}$$

In view of (29a) and (32), (26a) gives

$$T_0(r, \varphi, t) = \frac{a}{\pi A}\sum_{m=1}^{\infty}\left[\frac{J_0(\lambda_{0m} r)/J_0(\lambda_{0m} a)}{(a^2\lambda_{0m}^2 + \mu^2)}\int_0^{2\pi} f(\varphi', t)\,d\varphi' + 2\sum_{k=1}^{\infty}\frac{J_k(\lambda_{km} r)/J_k(\lambda_{km} a)}{(a^2\lambda_{km}^2 + \mu^2 - k^2)}\int_0^{2\pi} f(\varphi', t)\cos k(\varphi - \varphi')\,d\varphi'\right]. \tag{33}$$

To obtain a simple infinite series form of solution for $T_0(r, \varphi, t)$ we define a finite cosine transform of $T_0(r, \varphi, t)$ as follows:

$$\tilde{T}_0(r, k, t; \varphi') = \int_0^{2\pi} T_0(r, \varphi, t)\cos k(\varphi - \varphi')\,d\varphi, \tag{34}$$

the inversion of which is given by

$$T_0(r, \varphi, t) = \frac{1}{2\pi}\tilde{T}_0(r, 0, t; \varphi) + \frac{1}{\pi}\sum_{k=1}^{\infty}\tilde{T}_0(r, k, t; \varphi). \tag{35}$$

Applying the transform (34) to the system (27) we get

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{k^2}{r^2}\right)\tilde{T}_0(r,k,t;\varphi')=0, \qquad 0\leqslant r<a, \quad (36a)$$

with

$$A\frac{\partial\tilde{T}_0}{\partial r}(r,k,t;\varphi')+B\tilde{T}_0(r,k,t;\varphi')=\tilde{f}(k,t;\varphi'), \qquad r=a. \quad (36b)$$

Solution of (36) is readily obtained as

$$\tilde{T}_0(r,k,t;\varphi')=\frac{a}{A}\left(\frac{r}{a}\right)^k\frac{\tilde{f}(k,t;\varphi')}{(k+\mu)}. \quad (37)$$

Introducing (37) into (35) we have,

$$T_0(r,\varphi,t)=\frac{a}{2\pi A}\left[\frac{1}{\mu}\int_0^{2\pi}f(\varphi',t)\,d\varphi' + 2\sum_{k=1}^{\infty}\left(\frac{r}{a}\right)^k\frac{\int_0^{2\pi}f(\varphi',t)\cos k(\varphi-\varphi')\,d\varphi'}{(k+\mu)}\right]. \quad (38)$$

Comparison of (33) and (38) results in the summation formula:

$$\left(\frac{r}{a}\right)^k=2(k+\mu)\sum_{m=1}^{\infty}\frac{J_k(\lambda_{km}r)/J_k(\lambda_{km}a)}{(a^2\lambda_{km}^2+\mu^2-k^2)}, \qquad (\mu>0), \quad (39)$$

where the summation is over the positive roots of (30) for a given value of k. That (39) is indeed correct can be easily shown by expanding $(r/a)^k$ in an infinite series of $J_k(\lambda_{km}r)$ in $0\leqslant r<a$ and over m, and determining the expansion coefficients by utilizing the following integral formulas:

$$\int_0^a\left(\frac{r}{a}\right)^k J_k(\lambda_{km}r)\,r\,dr=\frac{(k+\mu)}{\lambda_{km}^2}J_k(\lambda_{km}a)$$

and

$$\int_0^a J_k^2(\lambda_{km}r)\,r\,dr=\frac{J_k^2(\lambda_{km}a)}{2\lambda_{km}^2}(a^2\lambda_{km}^2+\mu^2-k^2).$$

Although the expansion of an arbitrary function into a complete set of orthogonal functions is very easy, the inverse problem of finding the function to which a given infinite series of a complete set of orthogonal functions converges (in the mean) is not so easy, if at all possible. In other words, had we not utilized the above transform method of establishing (39), we might possibly have had difficulty in trying to sum the right-hand side of (39). The situation here is analogous to the ease of differentiation in contrast with the difficulty of integration of functions.

We have thus illustrated by this example how the $T_{0j}(P,t)$ functions of (11) in two-dimensional regions can be expressed in the form of simple infinite series rather than double infinite series. Similarly, the $T_{0j}(P,t)$ functions in three-dimensional regions can always be expressed by double-infinite series provided that their solution in the form of triple-infinite series is obtainable. In the case of one-dimensional regions this procedure yields the solutions of $T_{0j}(P,t)$ in closed form assuming again that they possess simple-infinite series form of solutions.

The boundary conditions (2) cover a wide variety of cases arising in engineering applications. In particular, conditions of prescribed surface temperature, prescribed surface heat flux and Newtonian boundary conditions, or any combination of these can be easily realized by assigning appropriate values to A_i, B_i and $f_i(s_i,t)$.

The general problem treated by Ojalvo [4] becomes a special case of the problem considered here. Indeed, when $Q(P,t)$ and $f_i(s_i,t)$ are separable functions of P and t, our solution (16) or (17) reduces to the combination of equations (4) and (19) together with equations (7'') and (23) integrated with respect to time, of the above reference.

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Résumé—Les techniques de transformation des intégrales finies sont appliquées à la solution des problèmes transitoires à trois dimensions de conduction de chaleur avec sources de chaleur fonctions générales du temps et des conditions aux limites. Ces dernières comprennent, comme cas speciaux, les conditions aux limites du premier, deuxième et troisième type (température donnée, flux de chaleur donné, conditions de convection de Newton) ou n'importe quelle combinaison de ces trois, telles que les ingénieurs les rencontrent souvent dans leurs problèmes.

La solution est obtenue à l'aide des termes en régime transitoire et en régime quasi-stationnaire et est donnée sous forme de séries infinies. D'après la solution, on voit comment le problème général peut être ramené à plusieurs problèmes plus simples pour lesquels on peut trouver facilement les solutions dans la litterature. Cet article constitute une généralisation d'un précédent article dans lequel on suppose que les fonctions représentant les sources de volume et de surface peuvent être séparées en variables espace et temps.

Zusammenfassung—Mit Hilfe der endlichen Integraltransformation werden dreidimensionale, instationäre Wärmeleitungsprobleme mit allgemein zeitabhängigen Wärmequellen und Grenzbedingungen gelöst. Letztere umfassen als spezielle Fälle die Grenzbedingungen erster, zweiter und dritter Art (vorgegebene Temperatur, vorgegebene Wärmestromdichte und Newtonscher Konvektionsansatz oder irgendeine Kombination dieser drei, wie sie oft in Ingenieurproblemen auftritt.

Die Lösung erhält man in Form quasistationärer und instationärer Ausdrücke, die als unendliche Reihen wiedergegeben sind. An Hand der Lösung wird gezeigt wie das allgemeine Problem auf verschiedene, einfachere reduziert werden kann, deren Lösungen gewöhnlich aus der Literatur zu erhalten sind. Die vorliegende Arbeit stellt eine Verallgemeinerung einer kürzlich erschienenen Arbeit dar, in der das Volumen und die Oberflächenquellenfunktionen als trennbar in Raum- und Zeitveränderliche angenommen waren.

Аннотация—Метод конечных интегральных преобразований использовался для решения трехмерных задач нестационарной теплопроводности при приозвольной зависимости от времени источников тепла и граничных условий. Последние включают как частные случаи граничные условия первого, второго и третьего рода (соответственно заданная температура, заданный тепловой поток и закон конвекции Ньютона) или любые комбинации этих трех условий, что часто встречается в технических задачах.

Решение выражается в виде квазистационарных и переходных членов и представлено в виде бесконечных рядов. Из этого решения видно, как можно общую задачу свести к нескольким более простым, решение для которых обычно легко найти в литературе. Статья обобщает данные одной из недавних статей, в которой предполагалось, что объемные и поверхностные части функций, описывающих источники, разделялись по пространственным и временным переменным.